# Monotonicity, Equilibrium Incentives, and Efficiency in a Dynamic Model of Hold up [*]

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**Revised, June 5, 2007**

## Abstract

We apply a framework in capital accumulation games to a dynamic context of Holdup, and perform a monotonicity analysis with the concept of Markov perfect equilibrium. The analysis reveals the existence of "monotone increasing first period incentives", "early turnpike" and "forecast horizon". These results show that for a sufficiently long horizon, first-period equilibrium incentives are insensitive to parameter changes *after* that horizon, namely, there is a *bounded planning horizon* in the sense that the sequences of parameters after that horizon (i.e., *far future contingencies*) do not affect the first-period (early-periods) equilibrium play. We interpret the results from the viewpoint of *contract incompleteness* that is crucial in the Holdup problems. We also prove with the monotone comparative statics method that in a more efficient accumulation system of relational-skill, more investments will be derived and higher efficiency will be dynamically attained in equilibrium.

*Key words*: A Dynamic Model of Hold up, Markov Perfect Equilibria, Turnpike, Forecast Horizon, Monotone Comparative Statics.

*JEL Classification*: C73, D23, L14

## 1. Introduction

In this paper, we construct an infinite-horizon model of Hold up, where the state variable is the relational skill at the beginning of a given period, and two players (Buyer and Seller) decide their individual investment levels taking into account their impact on strategic interactions from the next period onwards. We solve for a Markov Perfect Equilibrium of the game. Then, we introduce the outcomes of "Monotone Comparative Statics" à la Topkis (1978) and Milgrom and Roberts (1990) into the analysis of our dynamic model of "Hold up", which is modeled as a capital accumulation game concerning relation-specific skills.

Since Topkis (1978), a lot of research has focused on the analysis of monotonic behavior of optimal solutions and equilibrium outcomes, which are referred to as "Monotone Comparative Statics" (see Milgrom and Roberts (1990), and Edlin and Shannon (1998)). Referring to these powerful results, we show a turnpike property ("early turnpike" property) and prove the existence of a "forecast horizon" in a dynamic game of Hold up.

This paper attempts to make an original contribution to the literature of the dynamic models of Hold up, which Pitchford and Snyder (2004) is among the recently published papers on, by making use of the monotonicity properties and by showing a turnpike property and the existence of a forecast horizon in a dynamic model of Hold up. Our focus is the relationship between the monotonicity of optimal solutions and the existence of a forecast horizon, and our results show that, under certain conditions and for a sufficiently long horizon, first period (early periods) equilibrium incentives are insensitive to parameter changes *after* that horizon. In other words, there is a *bounded planning horizon* beyond which the sequence of parameters does not affect the equilibrium play in the first period (early periods). This also may be connected to an *endogenous* derivation of *contract incompleteness,* in that first-period equilibrium outcomes are insensitive to far future parameter changes (events), and so contracts do not need to reflect such future contingencies in their terms, that is, become non-state-contingent for such contingencies. This approach may provide a novel foundation for *contract incompleteness* crucial in the context of holdup problems.

For an applied contribution, it is conjectured that in a more efficient accumulation system of relational-skill, such as Toyota, more investments by Buyer and Seller will be derived, more relational-skill will be accumulated, and more efficiency (total surplus) will be attained in equilibrium. Our paper proves this conjecture in a simple and clear way, with the help of the monotone comparative statics method.

**Related Literature**

Konishi et al (1996) analyzed the Holdup problem and its solution through an endogenized tournament mechanism in a framework of Finite-Horizon continuous-time capital accumulation games.

For turnpike theorems in dynamic games, Ferstman and Kamien (1990) obtained the three (i.e., early, middle, and late) turnpike properties in a capital accumulation differential game with open loop strategies. On the other hand, we applied the framework of capital accumulation games to the context of Holdup, and then performed a monotonicity analysis with the concept of Markov perfect equilibrium. Thus, we obtained an "early turnpike" result, which says that the first-period (early periods) strategies in the original infinite horizon Markov perfect equilibrium are equal to those in a Markov perfect equilibrium in the finite horizon truncated game, when the truncated period is sufficiently far in the future.

For the literature on the dynamic model of Hold up, Pitchford and Snyder (2004) consider a setting in which the buyer's ability to hold up a seller's investment is so severe that there is no investment in equilibrium of the static game, and show that there exists an equilibrium of a dynamic game in which the seller makes a sequence of gradually smaller investments, each repaid by the buyer under the threat of losing future seller investment. The differences from our model are as follows. First, unlike our model, their model is not one of repeated transaction, but rather repeated offers between two players. Second, we adopt a Markov strategy rather than a "history dependent" strategy, and exclude the "punishment" mechanism they use to support the first best actions, in order to make heavy use of the monotonicity analysis in a dynamic game with the evolution of a state variable. Though it may be a rather technical treatment, our focus is different from theirs and indeed derives different main results and implications.

Thomas and Worrall (2005) investigate the conditions in which the first best efficiency can be attained in a self-enforcing way in an infinite repetition of a hold up model. Their model focuses on the self enforcement constraints to keep the players from deviating from the first best actions, though they also investigate the relationship between finite and infinite horizon games and present a convergence result as the time horizon goes to infinity. However, they do not consider the capital stock (relational skill) and its accumulation over time which plays an important role in our model, and indeed our main results are totally different in that we derive an early turnpike property, the existence of forecast horizon, and a new foundation for contract incompleteness.

Suzuki (2006) constructed an infinite-horizon model of Hold up where the state variable is the relational skill at the beginning of a given period, and analyzed a *stationary* Markov perfect equilibrium when the *stationarity* is imposed on the model. In contrast, this paper investigates the relationship between the infinite horizon game and the T- truncated finite horizon game of the model, as Fudenberg and Levine (1983) and Harris (1985) did, but with the help of a new method of monotone comparative statics. In that sense, Suzuki (2006) and this paper are companion papers.

Last, our paper is related also to the literature that analyze the extraction of a Common Property Resource (CPR), where two commercial fisheries at each period, fishing simultaneously in the same pool, exert a negative externality on each other, because the number of fish in the pool in the next fishing season depends on how many are left by the fisheries at the end of the current season, and usually lead to a socially inefficient fishing policy, i.e., the so-called *tragedy of the commons* (overexploitation).[1] Our model has an analogy to CPR in that given the relational skill at the beginning of the period, each of Buyer and Seller makes an investment, and exerts a *positive* externality with each other, which leads to *underinvestment*, i.e., the Holdup problem. The difference is that our model applies the monotone comparative statics method to a dynamic model of Holdup, rather than CPR, and derives the existences of early turnpike and forecast horizon, and compares the equilibrium of the game and the dynamic maximization of total surplus.

## 2. A Dynamic Model of Hold up

### 2.1 Set-up

We consider a dynamic game involving relation-specific skills and the Hold up problem. There are two fixed parties: Buyer B and Seller S. In each time period $t = 0, 1, 2, \ldots$, given the current level $x_t$ of relation-specific skill as a state variable, Buyer makes an investment $e_t^B$ which enhances valuation (revenue) $R\left(x_t, e_t^B\right)$ and Seller makes an investment $e_t^S$ which reduces cost $C\left(x_t, e_t^S\right)$, that is, $\partial R\left(x_t, e_t^B\right)\big/\partial e_t^B > 0$ and $\partial C\left(x_t, e_t^S\right)\big/\partial e_t^S < 0$. At the end of each period, the ex-post renegotiation surplus $R\left(x_t, e_t^B\right) - C\left(x_t, e_t^S\right)$ is generated. They renegotiate efficiently under symmetric information, and divide the renegotiation surplus 50/50 through the Nash Bargaining Solution. Hence, in each period, each player ex ante obtains half of the renegotiation surplus minus his own investment cost $e_i, i = B, S$.

Therefore in the stage game, given the level of relation-specific skill $x$, two players choose ex-ante investments as follows:

$$e_0^B \in \arg\max_{e^B} U^B\left(x, e^B, e^S\right) = \frac{1}{2}\left[R\left(x, e^B\right) - C\left(x, e^S\right)\right] - e^B$$

[1]Fudenberg and Tirole (1991pp510-512) review the literature on CPR in a dynamic game framework (e.g., Levhari and Mirman (1980), and Dutta and Sundaram (1993a), (1993b), which mainly prove existences of both symmetric and nonsymmetric Markov equilibria in general dynamic games.).

$$e_0^S \in \arg\max_{e^S} U^S\left(x, e^B, e^S\right) = \frac{1}{2}\left[R\left(x, e^B\right) - C\left(x, e^S\right)\right] - e^S$$

Then, we have *an underinvestment result*, since each party internalizes only 50% of its contribution to total surplus, while bearing all investment costs.

Now, the dynamics (the evolution of the state variable) in our game is modeled by

$$x_{t+1} = f\left(x_t + e_t^B + e_t^S\right) \qquad t = 0, 1, 2, \ldots$$

where $f(0) = 0$ and $f$ is monotonically increasing, concave, and continuously differentiable in the arguments. This formulation means that both players have access to the common specific skill $x_t$ at the beginning of time period $t$ and make investments $e_t^B, e_t^S$, and then the state in the next period $x_{t+1}$ is given by the above time-independent accumulation function. Moreover, there exists $\bar{x}_t > 0$ such that for every $x > \bar{x}_t$, we have $f(x) < x$. We assume that the state space at time period $t+1$ is $X_{t+1} \in [0, \bar{x}_t]$, regardless of investment levels $e_t^B$ and $e_t^S$. Moreover, let us assume that $M = \sup_t \bar{x}_t < \infty$ and we denote by $X = [0, M]$ the set of feasible states. Players have bounded maximum investment levels, and their action spaces at time period $t$ are continuous and compact, i.e., formally, $0 \leq e_t^i \leq K^i(x_t), i = B, S$.

The overall payoffs for the Infinite Horizon Game are the discounted sum of per-period payoffs:

$$\sum_{t=0}^{\infty} \delta^t U^i\left(x_t, e_t^B, e_t^S\right) \qquad i = B, S$$

where $\delta \in [0, 1)$ is a common discount factor, and we assume that the per period payoffs are uniformly bounded, i.e., there is a $Z$ such that

$$\sup_{i, t, e_t} U^i\left(x_t, e_t^B, e_t^S\right) \leq Z < \infty$$

Under these basic assumptions, the "*continuous at infinity*" condition as defined in Fudenberg and Levine (1983) and Fudenberg and Tirole (1991) will be satisfied.

## 2.2 Stage Game and Monotonicity of Optimal Solutions

We start with a brief review of the literature.

**Definition**

A function $\phi: E \times X \to \mathbb{R}$ where $E, X \subset \mathbb{R}$, has *Increasing Difference (ID)* if $\phi_e(e, x)$ exists

and it is weakly increasing in $x \in X$, for all $e$.

The intuition is that higher $x$ induces the marginal benefits of raising $e$. This property is also called *supermodularity*. A key result in monotone comparative statics is that when the objective function satisfies *Increasing Difference (ID)*, the maximizers are weakly increasing in the parameter value.

**Theorem** (Topkis (1978), also see Milgrom and Roberts (1990).)

Supposing that $\phi$ has *ID*, $x'' > x'$ and $E(x) = \arg\max\limits_{e \in E} \phi(e, x)$, then for any $e' \in E(x')$

and $e'' \in E(x'')$ we have $e'' \geq e'$.

We check the monotonicity of the equilibrium of our stage game. By assumption,

$\phi_B\left(x, e^B\right) = R\left(x, e^B\right)/2 - e^B$ and $\phi_S\left(x, e^S\right) = -C\left(x, e^S\right)/2 - e^S$ have an *ID*, that is:

$$x'' > x' \Leftrightarrow \frac{\partial \phi_B\left(x''\right)}{\partial e^B} = \frac{1}{2}R'\left(x'', e^B\right) - 1 \geq \frac{1}{2}R'\left(x', e^B\right) - 1 = \frac{\partial \phi_B\left(x'\right)}{\partial e^B} \quad \forall e^B$$

$$x'' > x' \Leftrightarrow \frac{\partial \phi_S\left(x''\right)}{\partial e^S} = -\frac{1}{2}C'\left(x'', e^S\right) - 1 \geq -\frac{1}{2}C'\left(x', e^S\right) - 1 = \frac{\partial \phi_S\left(x'\right)}{\partial e^S} \quad \forall e^S$$

Then, the payoff function $U^i\left(x, e^B, e^S\right), i = B, S$ satisfies the *ID* in $x$, since the marginal payoff

$\partial U^i / \partial e^i, i = B, S$ is monotonically non-decreasing in the parameter $x$. Hence, the best response

$BR^i\left(e^j, x\right), i, j = B, S, i \neq j$ is monotonically non-decreasing in $x$ for all $e^j$, and thus the

equilibrium is also monotonically non-decreasing in $x$ for all $p$. In this way, we obtain the

monotonicity of the equilibrium outcomes: $x'' > x' \Rightarrow e_{o,p}^i\left(x''\right) \geq e_{o,p}^i\left(x'\right), i = B, S$.

## 2.3 Single Crossing Property of the Dynamics

We assume that the dynamics is stationary and indexed by a parameter $p, q \in F$. Moreover, the indexing is such that it satisfies the following monotonicity property:

$$p > q \Leftrightarrow f^p(x) \ge f^q(x) \tag{1}$$

In words, *the higher the parameter, the higher the accumulation.* One interpretation is that the actual shape of the dynamics can vary for example, depending on whether or not the skill accumulation system is efficient. We make the convention whereby the *zero* parameter is associated with *zero* accumulation.

Now, we define the "Pointwise Gradient Order". Then, the "Pointwise Gradient *Monotone* Order"

is
$$f^p(x) \ge f^q(x) \implies \Delta_x f^p(x) \ge \Delta_x f^q(x) \tag{2}$$

where $\Delta$ is the first difference of $f(x)$ in $x$. We allow the differentiability of $f$.

We assume that two functions $f^p, f^q$ are ordered under *gradient monotonicity*. That is, $(1) \Rightarrow (2)$. This implies that the function $f$ satisfies *Increasing Difference (ID)*.

**Example** $p \in \{0.5, 1\}, f^p(x) = px^\alpha, 0 < \alpha < 1$

**Counterexample** $p \in \{0.5, 1\}, f^p(x) = \alpha x^p, 0 < \alpha < 1$

In the counterexample, there exists a $\hat{x}$ such that the "Pointwise Gradient Monotone Order" *reverses* pointwise. Conversely, the above example satisfies $(1) \Rightarrow (2)$, which is later required to obtain the monotonicity of the optimal solutions (e.g., the existence of monotone best responses), and so the monotonicity of (first period) equilibrium strategies $e_{0,p}^i(x_0), i = B, S$ in $p$.

**2.4 Finite Horizon Markov Perfect Equilibria**

A strategy for player $i = B, S$ is a sequence of maps of the form: $e_i^T = \left( e_0^i, e_1^i, e_2^i, \dots e_{T-1}^i \right)$

where $e_t^i : X_t \mapsto R^+$ .[2] We will restrict our attention to *"feasible"* strategies, i.e. $i = B, S$ :

$$e_t^i(x_t) \in \left[ 0, K^i(x_t) \right] \quad t = 0, 1, \dots, T-1$$

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[2] This is a *Markov* strategy in that strategies depend only on specified state variables $x_t$. See Fudenberg and Tirole (1991).

**<u>Definition</u>** *A pair of strategies* $\left(\overline{e}_B^T, \overline{e}_S^T\right) = \left(\overline{e}_t^B(x_t), \overline{e}_t^S(x_t)\right), t = 0,1,....,T-1$ *is called a **Markov***

***Perfect Nash Equilibrium** (MPE) of the dynamic game if **for every feasible state** $x_k \in X_k$ at time period $0 \le k < T$, we have for every feasible pair*

$$\left(e_B^T, e_S^T\right) = \left(e_t^B(x_t), e_t^S(x_t)\right), \ t = 0,1,..,T-1:$$

$$\sum_{t=k}^{T} \delta^t \ U^B\left(\overline{e}_t^B(x_t), \overline{e}_t^S(x_t)\right) \ge \sum_{t=k}^{T} \delta^t \cdot U^B\left(e_t^B(x_t), \overline{e}_t^S(x_t)\right)$$

$$\sum_{t=k}^{T} \delta^t \ U^S\left(\overline{e}_t^B(x_t), \overline{e}_t^S(x_t)\right) \ge \sum_{t=k}^{T} \delta^t \cdot U^S\left(\overline{e}_t^B(x_t), e_t^S(x_t)\right)$$

In summary, $\left(\overline{e}_B^T, \overline{e}_S^T\right)$ is said to be a MPE if and only if for every player $i = B, S$ at every state

$x_t$ at time period $t = 0,1,\cdots,T-1$, the player would find no incentive to deviate from the equilibrium strategies, as long as the other player follows them. In this equilibrium concept, the play to follow after every state $x_t$ prescribes a Nash equilibrium for the game that starts at $x_t$, which is commonly referred to as a *subgame*. In that sense, since the play off the equilibrium path is *credible*, this solution concept is time consistent. Hence, we can say that a MPE is a subgame perfect Nash equilibrium, where strategies depend only on specified state variables.

On the other hand, in Nash equilibrium of the dynamic game, each player $i = B, S$ commits himself to a future path once at the beginning of the game, and no player has an incentive to deviate by playing another feasible path from the initial state $x_0$, as long as the other player follows. However, the play prearranged after some state other than initial state $x_0$ may not constitute itself a Nash equilibrium for the subgame that starts at such a state. In other words, this solution concept is not time consistent (subgame perfect). Each player *ignores the evolution of the state variable* in the game, and does not optimally respond to each state $x_t$. It seems to be rather irrational. Thus, in order to avoid *non-credible equilibria* that may not prescribe equilibrium play after a subsequent state $x_t$, we use MPE as the equilibrium concept. [3]

Last, for an infinite horizon, the definition so far applies by simply defining the payoff function to be the liminf of the payoffs for the finite horizon truncation of the game, since the "one-stage-deviation principle" still holds under the "continuity at infinity" condition.

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[3] We chose a "Markov" strategy rather than a "history dependent" strategy to make heavy use of the monotonicity property. General (history dependent) strategies will generate the complexities of equilibrium and make the analysis messy. They do not harmonize well with *Monotone Comparative Statics* in our context.

## 2.5 Limit Theorem

For every $p \in F$, we shall denote by $\Pi_T^*(p)$ and $\Pi^*(p)$ the (possibly empty) set of MPE of the game indexed by $p$ for the first $T$−periods and the (possibly empty) set of MPE for the Infinite Horizon game indexed by $p$.

As Fudenberg and Tirole (1991) explained, the condition of 'continuity at infinity' assures that events after $t$ (for large $t$) have little effect, and the "one-stage-deviation principle" holds for finite- and infinite-horizon games. One could also expect that under the condition the sets of equilibria of finite-horizon and infinite-horizon versions of the 'same game' would be closely related, but it is not true that all infinite-horizon equilibria are limits of equilibria of the corresponding finite-horizon game. Hence, we need to formulate our standing assumptions, implied by the "Limit Theorem" of Fudenberg and Levine (1983), and also see Harris (1985).

**Assumption 1**: For every $p \in F_T$, any indexed collection $\left\{\left(\overline{e}_{T,p}^B, \overline{e}_{T,p}^S\right)\right\}_{T \in N}$ such that

$\left(\overline{e}_{T,p}^B, \overline{e}_{T,p}^S\right) \in \Pi_T^*(p)$, $\left(\overline{e}_{T,p}^B, \overline{e}_{T,p}^S\right) \to \left(\overline{e}_p^B, \overline{e}_p^S\right)$ as $T \to \infty$, where convergence occurs with

respect to the $FLH$ topology[4], we have $\left(\overline{e}_p^B, \overline{e}_p^S\right) \in \Pi^*(p)$.

We recall that two infinite horizon strategy pairs are close in the $FLH$ topology if they prescribe the <u>same</u> contingent plan of actions for <u>early stages</u> of the game. For complete studies on sufficient conditions that imply Assumption 1, see Fudenberg and Levine (1983), Harris (1985), and Fudenberg and Tirole (1991). Formally,

**Assumption 2**: For every $p \in F^\infty$, $\Pi^*(p)$ is compact with respect to the $FLH$ topology.

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[4]The notion of convergence related to the topology by FLH (Fudenberg-Levine (1983) and Harris (1985)) is the fact that $e_p^T \to e_p$ <u>if and only if</u> for all subgames, the sequence of histories induced by the $T$ horizon strategies $e_p^T$ simultaneously converges in the discrete topology (they <u>fully correspond</u>) to the histories induced by the infinite horizon strategies $e^p$. This convergence requires that for all $t$ there exists a $T(t)$ such that for all $T \geq T(t)$ the strategy combinations from the first period up to period $t$ as prescribed by $e_p^T$ and $e_p$ <u>correspond</u>.

## 3. Value Functions and Equilibrium Comparative Statics on the First Period Incentives

Let $V = \left(V^i\right)_{i=B,S}$ a $2-$tuple of value function $V^i : X \to R$ assigning a value to each state

$x$ of the game. First, we look at the player B's **recursive formulation** of his decision problem.

$$V_t^B\left(x_t; e_t^S, p\right) = \max_{e_t^B} U^B\left(x_t, e_t^B, e_t^S\left(x_t\right)\right) + \delta V_{t+1}^B\left(f^p\left(x_t + e_t^B + e_t^S\left(x_t\right)\right)\right)$$

where $V_t^B$ is the continuation value function for player $B$ at period $t$. In this problem, to

obtain *Increasing Difference (ID)* in $\left(e_t^B; p\right)$ for each $\left(e_t^S; p\right)$, the "envelopes" of the optimal

solutions of the players with respect to $p$ must be ordered pointwise Euclidean. To do this, we need

to investigate how a perturbation of $f$ (accumulation function) indexed by $p$ have effects on the

"envelopes" of $V_{t+1}^B$ in $x$ as $p$ increases. Given the differentiability of the value function

$V^i, i = B, S$, the first order condition for the maximization is given by:

$$\frac{1}{2}R'\left(x_t, e_t^B\right) - 1 + \delta V_{t+1}^{B\,\prime}\left(x_{t+1}\right) f^{p\prime}\left(x_t + e_t^B + e_t^S\left(x_t\right)\right) = 0 \qquad (3)$$

which gives us the function $e_t^B\left(x_t\right)$. Then, it follows from the "envelope theorem" that

$$\partial V_t^B / \partial x_t = dU^B\left(x_t, e_t^B, e_t^S\left(x_t\right)\right)/dx_t + \delta \cdot d V_{t+1}^B\left(f^p\left(x_t + e_t^B + e_t^S\left(x_t\right)\right)\right)/dx_t$$

$$= \frac{1}{2}\left[R'\left(x_t, e_t^B\right) - C'\left(x_t, e_t^S\left(x_t\right)\right)\right] - \frac{1}{2}C'\left(x_t, e_t^S\left(x_t\right)\right) \cdot \frac{de_t^S\left(x_t\right)}{dx_t}$$

$$+ \delta V_{t+1}^{B\,\prime}\left(f^p\left(x_t + e_t^B + e_t^S\left(x_t\right)\right)\right) \times f^{p\prime}\left(x_t + e_t^B + e_t^S\left(x_t\right)\right) \times \left(1 + e_t^{S\prime}\left(x_t\right)\right)$$

The last term in the right hand side of this equation shows that the current value of the state

variable $x_t$ affects the continuation value from the next period through its own increase in $x_t$ and

the other player $S$ 's investment level $e_t^S$ .

Now, suppose *hypothetically* that the current value of the state variable did not directly affect the

valuation from the next period so that the second term would disappear. That is, we have

$$\partial V_t^B / \partial x_t = dU^B\left(x_t, e_t^B, e_t^S\left(x_t\right)\right)/dx_t$$

$$= \frac{1}{2}\left[R'\left(x_t, e_t^B\right) - C'\left(x_t, e_t^S\left(x_t\right)\right)\right] - \frac{1}{2}C'\left(x_t, e_t^S\left(x_t\right)\right)\frac{de_t^S\left(x_t\right)}{dx_t}$$

which only captures the effects of the state variable on strategic interactions in the next period, in other words, the direct effect and strategic effect in the standard IO literature ala Tirole (1988). Then, letting $x_{t+1}$ denote the level of the state variable in the next period, we have from the above equations

$$\frac{1}{2}R'\left(x_t,e_t^B\right)-1+\delta V_{t+1}^{B\,\prime}\left(x_{t+1}\right)f^{p\,\prime}\left(x_t+e_t^B+e_t^S\left(x_t\right)\right)=0 \iff$$

$$\frac{1}{2}R'\left(x_t,e_t^B\right)-1+\delta f^{p\,\prime}\left(x_t+e_t^B+e_t^S\left(x_t\right)\right)$$

$$\times\left\{\frac{1}{2}\left[R'\left(x_{t+1},e_{t+1}^B\right)-C'\left(x_{t+1},e_{t+1}^S\left(x_{t+1}\right)\right)\right]-\frac{1}{2}C'\left(x_{t+1},e_{t+1}^S\left(x_{t+1}\right)\right)\frac{de_{t+1}^S\left(x_{t+1}\right)}{dx}\right\}=0$$

This is nothing but the best response in the Two-Period model. Again, the second term on the right hand side shows that the current value of the state variable $x_t$ affects the equilibrium value in the next period through its own increase in $x_{t+1}$ and the other player $S$ 's investment level $e_{t+1}^S$.

We see that the direct effect $\frac{1}{2}\left[R'\left(x_{t+1},e_{t+1}^B\right)-C'\left(x_{t+1},e_{t+1}^S\left(x_{t+1}\right)\right)\right]$ and the strategic effect

$-\frac{1}{2}C'\left(x_{t+1},e_{t+1}^S\left(x_{t+1}\right)\right)\frac{de_{t+1}^S\left(x_{t+1}\right)}{dx}$ are both positive, due to the assumption of the model and the *supermodurarity* of the stage game. Further we note that these direct and strategic effects do not exist in the "Static equilibrium" of the one-period game. Player S's problem can be analyzed similarly. Thus, we have the following proposition:

### Proposition 1: Equilibrium Incentives 1
*When the time horizon is Two-Period, the first period investments in the Markov Perfect Equilibrium are greater than those in the Static Equilibrium, due to the positive direct and strategic effects.*

Now, as the time horizon $T$ becomes longer, the dynamic effects consisting of the positive direct and strategic effects will be monotonically strengthened. Then, the dynamic *best response* for each player $i\neq j$ is monotonically increasing in the time horizon $T$ for all $e^j$, $x$, $p$, and thus the first period *equilibrium* is also monotonically increasing in the time horizon $T$ for all $x$, $p$. Thus, we obtain the monotonicity of the equilibrium incentives: $T'>T\Rightarrow e_{0,p}^{i,T'}\left(x_0\right)\geq e_{0,p}^{i,T}\left(x_0\right),i=B,S$

### Proposition2: Equilibrium Incentives 2

*For the equilibrium investments, we have the following monotone result.*

$$e_{0,p}^{i,1}(x) \le e_{0,p}^{i,2}(x) \le \cdots \le e_{0,p}^{i,T}(x) \le \cdots \le e_{0,p}^{i,\infty}(x), i = B, S$$

*That is, an increase in the horizon length T weakly increases the equilibrium investments in the first*

*period* (shown as a subscript $0$ ):  $T' > T \Rightarrow e_{0,p}^{i,T'}(x_0) \ge e_{0,p}^{i,T}(x_0), i = B, S$

This proposition can be understood as follows. Differentiating the first order condition (3) of the Bellman equation and putting it in order, we obtain

$$\frac{de_B}{de_S} = \frac{-\delta V^{B''} f^{p''}}{\frac{1}{2}R'' + \delta V^{B''} f^{p''}} > 0$$

This is due to the fact that the signs of the numerator and the denominator are both negative,

since $V_B'' < 0$, $f^{p''} < 0$, and the Second Order Condition $R''/2 + \delta V^{B''} f^{p''} < 0$ holds.[5] So, we

see that the first period strategies are *strategic complements*. That is, the dynamic reaction functions in the first period are *upward sloping* and the reduced form of the dynamic game played by these two players is "Super Modular". Then, by applying the Milgrom-Roberts (1990) Theorem, we have

$T' > T \Rightarrow e_{0,p}^{i,T'}(x_0) \ge e_{0,p}^{i,T}(x_0), i = B, S$ . The intuition is that the increase in time horizon $T$

shifts both players' (dynamic) best response curves upward, thus shifting the equilibrium upward. Based on these observations, we have the next result.

## Proposion3: Existence of 'Early Turnpike'

*There exists an infinite horizon Markov Perfect Equilibrium $\left(\overline{e}_p^B, \overline{e}_p^S\right)$ and a planning horizon $\overline{T}$*

*such that for $T \ge \overline{T}$ we have $e_{0,p}^{i,T}(x_0) = e_{0,p}^{i}(x_0)$. That is, we have an "Early Turnpike" result.*

## Proof: See Appendix1

Proposition3 states that a Markov perfect equilibrium in the infinite horizon game has an "early turnpike", which means that there exists a time $\overline{T}$ such that the first-period strategies in the

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[5] Increasing his own investment $e_i$ gives the positive externality to the other player in both Static and Dynamic frameworks, because $\partial u_{-i}(e_i, e_{-i})/\partial e_i > 0$ and $\partial V_{-i}/\partial e_i = \delta \cdot V_{-i}'' \cdot f'' > 0$.

original infinite horizon equilibrium are equal to those in a Markov perfect equilibrium in the T-truncated finite horizon game for all $T \geq \bar{T}$.

We examine the comparative statics on the change of parameter $p$. We can see that the increase in $p$ has *positive* effects on the "envelopes" of $V_{t+1}^B$ in $x$, because in the first order condition (3) of the Bellman equation:

$$\frac{1}{2} R'\left(x_t, e_t^B\right) - 1 + \delta V_{t+1}^{B'}\left(x_{t+1}\right) f^{p'}\left(x_t + e_t^B + e_t^S\left(x_t\right)\right) = 0$$

the increase in $p$ implies the *gradient monotonicity* in the sense of (2), which leads to the greater continuation value, and thus induces the greater equilibrium incentive in the first period.
This is exactly due to the *complementarities between $p$ and $x$ in the value functions*. This argument holds also for the player S's decision problem. Thus, we can order the gradients of the equilibrium function $e_0^i\left(x_0, p\right)$ for $i = B, S$ as $p$ changes. That is, the first period equilibrium incentives are monotonically increasing in $p$. We have that $p > q \Rightarrow e_{0,p}^i\left(x_0\right) \geq e_{0,q}^i\left(x_0\right)$ for $i = B, S$. Since the argument so far holds for any period length $T$, we have:


**Propositon4:** *There exists an infinite horizon Markov Perfect Equilibrium* $\left\{\left(\bar{e}_p^B, \bar{e}_p^S\right)\right\}_{p \in F}$ *such that the first period equilibrium incentives are monotonically increasing in $p \in F$, that is,*

$$p > q \Rightarrow e_{0,p}^i\left(x_0\right) \geq e_{0,q}^i\left(x_0\right), i = B, S.$$


Nonetheless, we find that for a sufficiently long horizon $T \geq \bar{T}$ the first period equilibrium incentives are not affected by future parameter changes even in the following specified way.

**Proposition5: Existence of 'Forecast Horizon'**

*There exists an infinite horizon Markov Perfect Equilibrium* $\left(\bar{e}_p^B, \bar{e}_p^S\right)$ *and a planning horizon* $\bar{T}$ *such that for $T \geq \bar{T}$ and every $q \in F$ with $p_t = q_t$ for $0 \leq t \leq \bar{T}$, we have*

$e_{0,p}^{i,T}\left(x_0\right) = e_{0,p}^i\left(x_0\right) = e_{0,q}^i\left(x_0\right)$. *Such a horizon is called **a "Forecast "horizon**.*


**Proof: See Appendix2**

Proposition5 states that a "forecast horizon" exists for <u>some</u> Markov perfect equilibrium which means that there exists a time $\overline{T}$ such that the first-period strategies in the original infinite horizon equilibrium are equal to those in a Markov perfect equilibrium in the T truncated finite horizon game for all $T \geq \overline{T}$, regardless of parameter values affecting periods beyond $\overline{T}$. Propositions 3 and 5 can be interpreted as *approximation results*.

**4. Efficiency**

In this section, we apply the Milgrom-Roberts theorem (1990) to our dynamic game model and perform an analysis on the dynamic efficiency.

Let $\phi_B\left(x_0, e_0^B, e_0^S\right) = U_B\left(x_0, e_0^B, e_0^S\right) + \delta V_B\left(f\left(x_0 + e_0^B + e_0^S\right)\right)$ be the continuation payoff

function for player B, which is divided into the today's payoff $U_B\left(x_0, e_0^B, e_0^S\right)$ and the tomorrow's

(future) payoff $V_B\left(f\left(x_0 + e_0^B + e_0^S\right)\right)$. We assume that $V_B\left(f\left(x_0 + e_0^B + e_0^S\right)\right)$ is the

continuation value which is optimized in equilibrium given the state $x_1 = f\left(x_0 + e_0^B + e_0^S\right)$.

$\phi_S = U_S + \delta V_S$ is defined similarly. Then, $S = \phi_B + \phi_S$ represents a dynamic total surplus

(efficiency), and we obtain the following proposition.

<u>**Proposition 6**</u>**: Efficiency**

Dynamic efficiency (total surplus) is weakly increasing in parameter $p > q$, i.e., $S(q) \leq S(p)$

for $q < p$, but still less than the dynamic first best efficiency.

<u>**Proof**</u>**: See Appendix3**

Suzuki (2006) presents a conjecture that *in the stationary Markov Perfect Equilibrium, the equilibrium incentives are monotonically increasing in $p$*. One interpretation is that we can view $p$ as an efficient skill accumulation system, such as in Toyota, and $q$ as another less efficient one, and that as the accumulation of relational skill is more efficient, that is $p > q$, the equilibrium investments and the relational skill will become greater in the stationary Markov Perfect Equilibrium. Propositions 4 and 6 mean a closer examination of Suzuki (2006)'s conjecture.

## 5. Concluding Remarks

This paper applied the framework of capital accumulation games to a dynamic context of Holdup, and then performed a monotonicity analysis using the concept of Markov perfect equilibrium. As a result, we obtained an "early turnpike" result, while Ferstman and Kamien (1990) obtained the three (i.e., early, middle, and late) turnpike properties in a capital accumulation differential game with open loop strategies. We also obtained the existence result of a "forecast horizon".

Interestingly enough, the existence of such horizon indicates that "backward induction" can be performed for infinite horizon games, in the sense that, to solve for the first-period equilibrium, one may consider the finite horizon dynamic game whose planning horizon is exactly the forecast horizon. The existence of a "forecast horizon" implies that for a sufficiently long horizon, we can obtain *strategic decoupling,* i.e. first-period equilibrium incentives are insensitive to parameter changes *after* that horizon. In other words, there is a *bounded planning horizon* after which the sequence of parameters does not affect the equilibrium play in the first period. This could be said to be a different derivation from Jehiel (1995) of the *limited forecast horizon*.

Also in a sense, this result could also be said to be an endogenous derivation of *contract incompleteness*, in the sense that first-period equilibrium incentives become *insensitive to far future parameter changes (events),* and so, contracts do not need to reflect such future contingencies in their terms, that is, become non-state-contingent for such contingencies.[6] There is some existing literature on the foundation for contractual "incompleteness". Segal (1999) defines a notion of *complexity* of the trading environment, and shows that when *complexity* grows without bound, there is asymptotically no value of contracting, in a specific Hold up model.[7] On the other hand, our paper shows that when horizon length $T$ grows, contracting loses its power for all periods beyond $\bar{T}$, due to the players' *insensitiveness to* far future parameter changes (events).[8] Hence, our model may provide a new foundation for *contract incompleteness* that is crucial in the context of Hold up problems.

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[6]In the second period $t = 1$, again, it is enough that the "renegotiated" contract can cover $\bar{T}$ duration, since players only foresee up to $\bar{T} + 1$ period. It again becomes non-state-contingent for the contingencies beyond $\bar{T} + 1$. This argument applies up to $T - \bar{T}$ periods, given $T$.

[7] See Bolton and Dewatripont (2005) for a survey of this topic.

[8]Hart and Moore (1999), who provides a more tractable model than Segal (1999), interprets contractual incompleteness, and mentions that it seems necessary to assume that the parties are bounded rational in the sense that they do not foresee even relatively obvious events. It is of course different from our dynamical viewpoint.

# APPENDICES

## <u>Appendix1</u> **Proof of Proposition 3**

Let us now append <u>the *zero* strategy</u> to $\left(\overline{e}^B_{T,p}, \overline{e}^S_{T,p}\right)$ at period $T$. i.e.:

$$\left(e^{i,T}_{0,p}, e^{i,T}_{1,p}.....e^{i,T}_{T-1,p}, 0\right)$$

By the definition of MPE, we have that the extended pair must be an MPE to the $T+1-$horizon game with parameters $\underbrace{(p, p,..., p}_{T\ \text{times}}, 0)$ and by the definition of the *zero* parameter:

$$\underbrace{(p,..., p}_{T\ \text{times}}, 0) \leq \underbrace{(p,..., p}_{T\ \text{times}}, p)$$

hence, by *the monotonicity*, it follows that:

$$e^{i,T}_{0,p}(x_0) \leq e^{i,T+1}_{0,p}(x_0)$$

,the micro foundation of which has been presented above.

Now, by the compactness, the sequence $\left\{e^{B,T}_{0,p}(x_0), e^{S,T}_{0,p}(x_0)\right\}_{T\in N}$ must converge.

On the other hand, by assumption 1 (Fudenberg and Levine 1983, Harris 1985), the infinite extension: $\qquad \left(e^{i,T}_{0,p}, e^{i,T}_{1,p}.....e^{i,T}_{T-1,p}, 0, 0...\right) \in \Pi^* \underbrace{(p, p,..., p}_{T\ \text{times}}, 0, 0,..)$

Moreover, by assumption 2 (compactness of the set of MPE for the infinite horizon games: Fudenberg and Levine 1983, Harris 1985), the extended collection has a converging subsequence.

Let us denote by $e^i_p$ the limit of such subsequence:

$$\lim_{k\to\infty} e^{i,T_k}_p = e^i_p$$

Now by convergence in the *FLH* topology:

$$\lim_{T\to\infty} e^{i,T}_{0,p} = e^i_{0,p}$$

Or equivalently: there exists a planning horizon $\overline{T}$ such that for $T \geq \overline{T}$ we have[9]:

$$e^{i,T}_{0,p}(x_0) = e^i_{0,p}(x_0) \qquad\qquad \textbf{Q.E.D}$$

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[9]Note that this is the result on the first (early) period equilibrium outcome, and that we imposed more restrictive conditions on the convergence. That is why we have the "<u>exact</u>" result. Otherwise, we could only state that *there exists an infinite horizon Markov Perfect Equilibrium* $\left(\overline{e}^B_p, \overline{e}^S_p\right)$ *such that* for every $\varepsilon > 0$, *there exists a planning horizon* $\overline{T}$, *such that for* $T \geq \overline{T}$ *we have (in an <u>approximate</u> form):* $\left|e^{i,T}_{0,p}(x_0) - e^i_{0,p}(x_0)\right| < \varepsilon$.

## Appendix2 Proof of Proposition 5

Let us construct a sequence of parameters based on $p \in F$ as follows; we append a minimum

forecast[10] $\underline{p} = 0$ to the $T-$ truncation of $p = (p, p, ...)$, and obtain the forecast:

$$u(T) = (\underbrace{p, p, ...., p}_{T \text{ times}}, 0, 0...)$$

where clearly we have: $\qquad u(T) \to p \quad \text{as} \quad T \to \infty$

But by the Early Turnpike Theorem (Theorem1), we know that:

$$\lim_{T \to \infty} \left| e_{0,p}^{i,T} - e_{0,p}^{i} \right| = 0 \qquad\qquad (4)$$

and by the analysis so far, the sequence $\left\{ e_0^{u(T)} \right\}$ is monotonically increasing in $T$, i.e.:

$$e_0^{u(T)} \leq e_0^{u(T+1)}$$

By compactness of the first period strategy space and continuity of the map $e_{0,p}^{i} : F \mapsto R^{+}$

$$\lim_{T \to \infty} \left| e_{0,u(T)}^{i} - e_{0,p}^{i} \right| = 0 \qquad\qquad (5)$$

So the results (4) and (5) ensure the existence of a large enough horizon $\overline{T}$ such that for $T \geq \overline{T}$ :

$$e_{0,p}^{i,T} = e_{0,u(T)}^{i} = e_{0,p}^{i}$$

Now let us consider $q \in F$ such that $p_t = q_t$ for $0 \leq t \leq \overline{T}$ , i.e., $q = (\underbrace{p, p, ...., p}_{\overline{T} \text{ times}}, q, q, ...)$. Then,

by *monotonicity*, it follows that for the choice of $\overline{T}$, for every $T \geq \overline{T}$, $e_{0,u(\overline{T})}^{i} \leq e_{0,q}^{i} \leq e_{0,p}^{i}$

Hence, we have $e_{0,p}^{i} = e_{0,q}^{i}$ . In words, there exists an infinite horizon first period equilibrium

outcome that is *insensitive to parameter changes after time period* $\overline{T}$ \qquad\qquad **Q.E.D**

## Appendix3 Proof of Proposition 6

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[10] By assumption, $F$ is a finite set endowed with a particular ordering $\geq$ , according to which there is a

minimum $\underline{p}$ and a maximum forecast $\overline{p}$ . We assume that $\underline{p} = 0$.

First change the accumulation parameter from $q$ to $p$ while holding investments fixed at $e_0(q)$. The first period total surplus does not change. Then release investments----they go up. Each player $i$ is better off because the increase in $e^{-i}$ has a positive externality on him, and his increase in $e^i$ is voluntary (incentive compatible, self-enforcing). Formally,

$$\phi_i\left(e_0^B(q), e_0^S(q), p\right) \le \phi_i\left(e_0^B(q), e_0^S(p), p\right) \le \phi_i\left(e_0^B(p), e_0^S(p), p\right), \text{ for each } i = B, S,$$

where the first inequality is due to positive externalities and the second is due to the first period equilibrium condition. Adding over $i = B, S$, we have

$$S\left(e_0^B(q), e_0^S(q); q\right) \le S\left(e_0^B(p), e_0^S(p); p\right).$$

This completes the proof of the former part of the lemma.

For the latter part, we similarly introduce a parameter $\theta \in \{0, 1\}$ and construct a dummy objective function $T_i(e, x, \theta) = \phi_i\left(x, e^i, e^{-i}\right) + \theta \phi_{-i}\left(x, e^i, e^{-i}\right)$, where

$$T_i(e, x, \theta) = \begin{cases} \phi_i\left(x, e^i, e^{-i}\right) \text{ when } \theta = 0 \\ S\left(x, e^i, e^{-i}\right) \text{ when } \theta = 1 \end{cases}$$

Whether $\theta = 0$ or $\theta = 1$ means whether player $i$ takes into consideration the positive externality to another player. Now, we easily check that

$$\partial \phi_{-i}\left(x, e^i, e^{-i}\right) \big/ \partial e^i = \underbrace{\partial u_{-i} \big/ \partial e^i}_{+} + \delta \cdot \underbrace{\partial V_{-i} \big/ \partial e^i}_{+} > 0, \text{ where } \phi_{-i}\left(x, e^i, e^{-i}\right) = u_{-i} + \delta V_{-i}$$

This in turn implies that $T_i(e, x, \theta)$ has ID in $\left(e^i, \theta\right)$. Then, by applying the Milgrom-Roberts (1990) Theorem, we have $e_0(\theta = 1) \ge e_0(\theta = 0)$. This implies that the solution to the second objective function (dynamic "social optimum" equilibrium) is greater than the solution to the first (dynamic "private optimum" equilibrium, i.e., dynamic "Hold up" equilibrium). That is,

$$e^{FB}(x) \ge e_0(x). \qquad\qquad\qquad\qquad\qquad \textbf{Q.E.D}$$

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